JOURNAL OF APPROXIMATION THEORY 74, 1–15 (1993)

A Dynamical Systems Approach to the Polygonal Approximation of Plane Convex Compacts*

PETAR S. KENDEROV AND NIKOLAI K. KIROV

Institute of Mathematics with Computer Center, Bulgarian Academy of Sciences, Sofia, Bulgaria

Communicated by Alphonse P. Magnus

Received July 2, 1990; accepted in revised form March 11, 1992

A connection is described between the polygonal approximation of a compact convex set in R^2 and some dynamical systems on the unit circumference in R^2. Based on this a numerical procedure is proposed for finding a best approximating n-gon for an arbitrary compact convex set in R^2 (w.r.t. Hausdorff metric). The algorithm provides a solution to a specific global optimization problem where the function to be minimized has more than one local minimum. In one of its equivalent formulations the above approximation problem can be considered as a specific spline approximation problem. From this point of view our algorithm provides also a solution to a specific variable knots spline approximation problem.

1. INTRODUCTION

Let R^2 be the usual plane with the Euclidean norm $|\cdot|$ and let CONV be the set of all convex compact subsets of R^2. In CONV we consider the so-called Hausdorff metric which is defined by the formula $h(A_1, A_2) = \inf\{t > 0 : A_1 \subset A_2 + tB,\ A_2 \subset A_1 + tB\}$ where $B = \{P \in R^2 : |P| \leqslant 1\}$ is the unit circle in R^2, $C_1 + C_2 = \{P_1 + P_2 : P_i \in C_i, i = 1, 2\}$ is the Minkowski sum of sets C_1, C_2, and $tB = \{tP : P \in B\}$. For a given integer $n \geqslant 3$ we denote by POLY_n the set of all convex polygons with not more than n vertices. For a given set A the n-gon $\Delta_0 \in \mathrm{POLY}_n$ will be called a best Hausdorff approximation of A in POLY_n if $h(A, \Delta_0) = \inf\{h(A, \Delta) : \Delta \in \mathrm{POLY}_n\}$. The existence of at least one best approximation for A (in any POLY_n) is a corollary of the well-known Blaschke "selection theorem" asserting that every bounded sequence of sets from POLY_n (n-fixed) contains a subsequence converging in the Hausdorff metric to some n-gon. In general, however, the best approximation is not unique.

* This work was partially supported by Grants 263/87 and 58/91 of the Bulgarian Ministry of Science and Higher Education.

0021-9045/93 $5.00

We show here that there exists a natural connection between the above approximation problem and some dynamical system on the unit circumference S of R^2 (under "dynamical systems" we understand here a family of homeomorphisms of S into itself). Based on this connection we give here an algorithm for finding a best approximation in POLY_n for an arbitrary $A \in \mathrm{CONV}$.

Our procedure contains several steps. Given the set $A \in \mathrm{CONV}$, we first associate with each number $\varepsilon > 0$ some order preserving homeomorphism T_ε of S onto itself. This is done in such a way that the rotational number $r(\varepsilon)$ of T_ε is a nondecreasing function of $\varepsilon > 0$ and, moreover, the number $\varepsilon_n^* := \min\{\varepsilon > 0: r(\varepsilon) = 1/n\}$ is equal to the distance between A and its best approximations in POLY_n.

In the second step we take an arbitrary unit vector $e \in S$ and generate the iterations

$$T_\varepsilon(e),\ T_\varepsilon^2(e),\ \ldots,\ T_\varepsilon^k(e),\ \ldots \qquad \text{for} \quad \varepsilon = \varepsilon_n^*.$$

It turns out that this sequence approximates very well some n-periodical sequence $\{e_k\}_{k \geqslant 1} \subset S$ such that all e_k, $k = 1, 2, \ldots, n$ are "outward" unit normals to the sides of some polygon Δ which is a best approximation for A in POLY_n. This suffices to construct (in the third step) the best approximating polygon $\Delta \in \mathrm{POLY}_n$.

In one of its equivalent formulations the above problem can be considered as a specific spline approximation problem. From this point of view our algorithm provides a solution to a specific variable knots spline approximation problem. It also gives a solution to a particular global optimization problem where the function to be minimized has more than one local minimum.

Some other aspects of the polygonal approximations of plane convex sets are contained in the papers of Popov [P], Toth [T], McLure and Vitale [McV], Georgiev [Ge], Gruber and Kenderov [GK], Nedelcheva [Ne], etc.

2. Necessary Results from Convexity and the Theory of Dynamical Systems

We denote the scalar product of two vectors (points) P_1, P_2 in the plane R^2 by $\langle P_1, P_2 \rangle$. Thus the Euclidean norm of some $P \in R^2$ is $|P| = \sqrt{\langle P, P \rangle}$. For a given $A \in \mathrm{CONV}$ we denote by s_A the support function of A defined in R^2 by the formula $s_A(P) = \max\{\langle P, X \rangle : X \in A\}$. This function is convex, positively homogeneous, and continuous. It is completely determined by its values on the unit circumference $S := \{P \in R^2: |P| = 1\}$. For every A, A_1, $A_2 \in$ CONV and $t>0$ we have $s_{A_1+A_2}=s_{A_1}+s_{A_2}$ and $s_{tA}=ts_A$. Moreover $A_1 \subset A_2$ iff $s_{A_1}(e) \leqslant s_{A_2}(e)$ for every $e \in S$. These facts, together with the observation that the support function of $B=\{P \in R^2: |P| \leqslant 1\}$ on S is the constant 1, show that the Hausdorff distance between two sets $A_1, A_2 \in$ CONV can be represented as follows: $h(A_1, A_2)=\inf\{t>0: s_{A_1}(e) \leqslant s_{A_2}(e)+t, s_{A_2}(e) \leqslant s_{A_1}(e)+t$ for every $e \in S\}=\inf\{t>0: |s_{A_1}(e)-s_{A_2}(e)| \leqslant t$ for every $e \in S\}=\max\{|s_{A_1}(e)-s_{A_2}(e)|: e \in S\}$. This means that the mapping assigning to each $A \in$ CONV the function s_A from the space $C(S)$ of all continuous functions in S is an isometry when CONV is given the Hausdorff metric and $C(S)$ is equipped with the usual "sup" norm. Having this in mind we see that the problem of approximating an $A \in$ CONV (with respect to the Hausdorff distance) by elements of POLY$_n$, $n \geqslant 3$, is equivalent to the approximation (in the sup-norm in $C(S)$) of s_A by support functions of n-gons.

Let $\Delta \in$ POLY$_n$ be an n-gon with different vertices $P_1, P_2, ..., P_n$ (taken in the counterclockwise direction). The so called "side directions" of Δ are the unit vectors e_i, $i=1, 2, ..., n$, such that e_i is perpendicular to the side $\overline{P_iP_{i+1}}$ of Δ ($P_{n+1} \equiv P_1$) and is directed "outward Δ." Of course, $s_\Delta(e)=\max\{\langle P_i, e\rangle: i=1, 2, ..., n\}$ and $s_\Delta(e_i)=\langle P_i, e_i\rangle=\langle P_{i+1}, e_i\rangle$, $i=1, 2, ..., n$. On the arc $[e_i, e_{i+1}] \subset S$ (taken again in the counterclockwise direction) $s_\Delta(e)=\langle P_{i+1}, e\rangle$. In this sense $s_\Delta(e)$ is a "spline" of functions of the type $\langle P, e\rangle$ (P is fixed and e is the variable) with knots at the side directions e_i, $i=1, 2, ..., n$. If S is identified with the segment $[0, 2\pi]$, then $s_\Delta(\varphi)$ is a spline of functions of the type $a\cos(\varphi-\alpha)$ where a and α are fixed.

For a given $M \in R^2$ and $A \in$ CONV we denote by $d(M, A)$ the distance from M to A, i.e., $d(M, A)=\min\{|M-X|: X \in A\}$. If $M \notin A$ then $d(M, A)>0$ and there exists a uniquely determined "nearest point" $N=N(M)$ for M in the set A: $N \in A$ and $|M-N|=d(M, A)$.

Definition 2.1. The n-gon $\Delta=(P_1, P_2, ..., P_n)$ is said to be alternating for the set $A \in$ CONV if for each $i=1, 2, ..., n$

(a) $d(P_i, A)=h(A, \Delta)$,

(b) $s_A(e_i)-s_\Delta(e_i)=h(A, \Delta)$.

If $N_i=N(P_i)$ is the nearest point in A for P_i, $i=1, 2, ..., n$, and $e_i^* := (P_i-N_i)/d(P_i, A)$, then the unit vectors $e_1^*, e_1, e_2^*, e_2, ..., e_n^*, e_n$ appear one by one in the counterclockwise direction on S and for each $i=1, 2, ..., n$, $h(A, \Delta)=s_A(e_i)-s_\Delta(e_i)=s_A(e_i^*)-s_\Delta(e_i^*)$. This explains the notion "n-gon alternating for A." The 3-gon $\Delta=(P_1, P_2, P_3)$ in Fig. 2.1 is alternating for the set A.

The following theorem (see Theorem 3.1 in [Ke]) shows the role played by the alternating n-gons in the polygonal approximation of a given $A \in$ CONV.

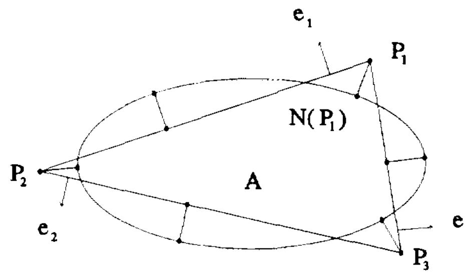

FIGURE 2.1

THEOREM 2.2. *If Δ is a best Hausdorff approximation in* POLY_n *for some* $A \in \mathrm{CONV}$, *then Δ is alternating for A.*

Thus, the alternating property is a necessary condition for Δ to be a best approximation for A in POLY_n. However, unlike the Čebishev approximation by polynomials, this condition is very far from being a sufficient one. The next assertion shows that there are many n-gons alternating for a given A and, in general, their distance from A is not one and the same.

THEOREM 2.3 (see 4.10 from [Ke]). *Let* $A \in \mathrm{CONV} \setminus \mathrm{POLY}_n$ *and* $e \in S$. *Then there exists a unique n-gon* $\Delta_n(e) \in \mathrm{POLY}_n$ *which is alternating for A and has e among its side directions.*

In fact (see 4.12 from [Ke]), $\Delta_n(e)$ provides the best Hausdorff approximation for A in the set of all elements of POLY_n having e among their side directions.

Since there is an efficient algorithm (see Sect. 3 of this paper) for finding $\Delta_n(e)$, Theorems 2.2 and 2.3 reduce our problem of finding best Hausdorff approximation for A in POLY_n to the minimization of the function $t_n(e) := h(A, \Delta_n(e))$ over $e \in S$. This approach was already used by Yotov and Christov [YC] for the partial case when A is a polygon with more than n vertices. However, to minimize $t_n(e)$ over $e \in S$ is not a trivial task. There are at least two kinds of difficulties. As seen from Fig. 2.2 (a), (b), and (c) where the graphs of $t_n(e)$ for different A's and n's are plotted, the function $t_n(e)$ can have many local minima and this is an obstacle for the numerical optimization.

Another obstacle is the fact that the function $t_n(e)$ is not obliged to be everywhere differentiable and the standard optimization procedures can not be applied.

In this paper we suggest a way to avoid these difficulties. It is based on some simple considerations from the theory of dynamical systems in S.

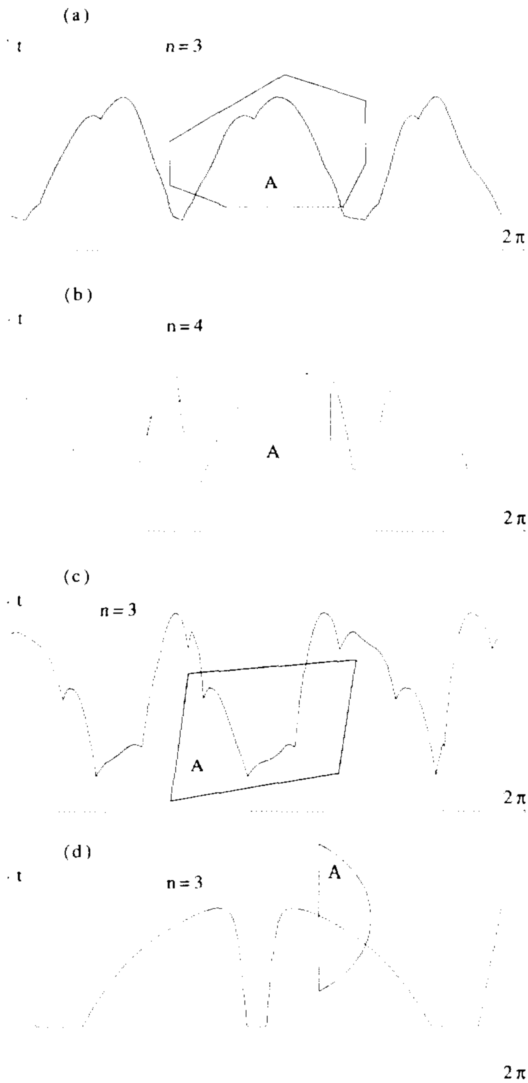

FIG. 2.2. Graph of functions $t_n(e)$. Corresponding sets A are also depicted; the unit vectors $e \in S$ are identified in a natural way with the real numbers from $[0, 2\pi]$.

Construction 2.4. Let $A \in \text{CONV}$ have interior points. Let $\varepsilon > 0$. We define now a mapping (dynamical system) $T(\cdot, \varepsilon)$: $S \to S$ in the following way. For $e \in S$ consider the line (see Fig. 2.3) $L = \{X \in R^2$: $\langle e, X \rangle = s_A(e) - \varepsilon\}$ and a point $P(e) \in A + \varepsilon B$ (the ε-neighborhood of A) for which $s_A(e) + \varepsilon = \langle P(e), e \rangle$. The intersection of L and $A + \varepsilon B$ is a segment

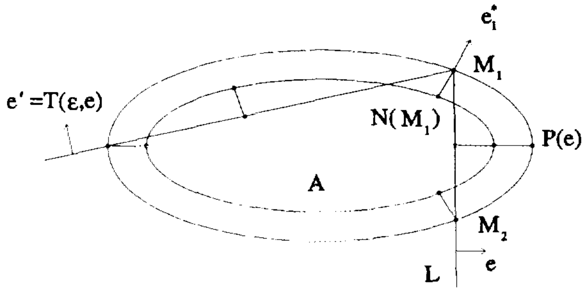

FIGURE 2.3

$\overline{M_1 M_2}$. Without loss of generality we may consider that the point M_1 "comes first" when the boundary of $A + \varepsilon B$ is run over in the counter clockwise direction starting from $P(e)$ (this means that the interior of $A + \varepsilon B$ remains always on the left-hand side). Denote by e_1^* the unit vector $(M_1 - N(M_1))/\varepsilon$ where, as above, $N(M_1)$ is the nearest point in A for M_1. Further, denote by $T(e, \varepsilon)$ the only unit vector $e' \in S$ for which $s_A(e') - \varepsilon = \langle e', M_1 \rangle$ and $e' \in [e_1^*, -e_1^*]$ (the arc $[e_1^*, -e_1^*]$ is taken in counterclockwise direction).

The correctness of this construction is easily seen from the following fact (see Proposition 2.1 of [Ke]):

For every M_1 with $d(M_1, A) = \varepsilon$ the function $s_A(e) - \langle e, M_1 \rangle$ strictly increases (when e runs over the arc $[e_1^*, -e_1^*]$) from $-d(M_1, A) = -\varepsilon$ (for $e = e_1^*$) to $d(M_1, A) = \varepsilon$ for some $e = e'$ and takes values greater than ε in the open arc $(e', -e_1^*)$. Thus the point $M_1 = M_1(\varepsilon, e)$ and the unit vectors $e_1^* = e_1^*(e, \varepsilon)$ and $T(e, \varepsilon)$ are completely determined by the following conditions:

(a) $s_A(e) - \langle e, M_1 \rangle = s_A(T(e, \varepsilon)) - \langle T(e, \varepsilon), M_1 \rangle = d(M_1, A)$;

(b) $e_1^* \in [e, -e]$;

(c) $T(e, \varepsilon) \in [e_1^*, -e_1^*]$.

The length of the arc $[e, T(e, \varepsilon)]$ (again in counterclockwise direction) is strictly between 0 and 2π.

In the special case when A is a circle of radius r, $T(\cdot, \varepsilon)\colon S \to S$ is just the rotation in the counterclockwise direction by the angle $2 \arccos((r - \varepsilon)/(r + \varepsilon))$.

Denote by R^+ the set of all positive real numbers. For us the important properties of the mapping $T(\cdot, \cdot)\colon S \times R^+ \to S$ assigning to each pair $(e, \varepsilon) \in S \times R^+$ the element $T(e, \varepsilon) \in S$ are gathered in the following assertion.

PROPOSITION 2.5. (i) *T is continuous at every point $(e_0, \varepsilon_0) \in S \times R^+$.*

(ii) *If $\varepsilon_1 > \varepsilon_2$, then for every $e \in S$ $T(e, \varepsilon_2)$ belongs to the open arc $(e, T(e, \varepsilon_1))$ (again taken in the counterclockwise direction).*

(iii) *For a fixed ε the mapping $T(\cdot, \varepsilon)$: $S \to S$ is one-to-one, onto S, and "preserves the order," i.e., if $e_2 \in (e_1, T(e_1, \varepsilon))$ then $T(e_1, \varepsilon) \in (e_2, T(e_2, \varepsilon))$.*

This is contained in the proof of Proposition 2.7 from [Ke].

When $\varepsilon > 0$ is fixed, the behavior of the sequence of iterations

$$e, T(e, \varepsilon), T^2(e, \varepsilon), ..., T^n(e, \varepsilon), ...,$$

where $e \in S$, $T^{k+1}(e, \varepsilon) = T(T^k(e, \varepsilon), \varepsilon)$, and $T^0(e, \varepsilon) := e$, is of special importance for us. For instance, if the sequence is periodical (with minimal period k) and makes only one turn around S for k steps, then the sequence $\{T^n(e, \varepsilon)\}_{n \geqslant 1}$ consists of side directions of some k-gon Δ which is alternating for the set A and $h(A, \Delta) = \varepsilon$. It is easy to find the vertices of Δ in this case. For every pair of successive side directions $e' = T^i(e, \varepsilon)$ and $e'' = T^{i+1}(e, \varepsilon)$ the intersection point of the lines $L' = \{P \in R^2: \langle e', P \rangle = s_A(e') - \varepsilon\}$ and $L'' = \{P \in R^2: \langle e'', P \rangle = s_A(e'') - \varepsilon\}$ is a vertex of Δ.

One could not expect that for every $e \in S$ the sequence $\{T^n(e, \varepsilon)\}_{n \geqslant 0}$ will be k-periodical. However, the following statement is true.

COROLLARY 2.6. *If there exists an alternating k-gon Δ_0 for the set $A \in$ CONV with $h(A, \Delta_0) > 0$, then for every $e \in S$ the sequence $\{T^n(e, \varepsilon)\}_{n \geqslant 0}$, where $\varepsilon = h(A, \Delta_0)$, will be "asymptotically" k-periodical. This means that $\{T^{kn}(e, \varepsilon)\}_{n \geqslant 0}$ is a convergent subsequence and for $\bar{e} = \lim_{n \to \infty} T^{kn}(e, \varepsilon)$ the sequence $\{T^n(\bar{e}, \varepsilon)\}_{n \geqslant 0}$ is k-periodical and consists of side directions of some k-gon $\bar{\Delta}$ which is alternating for A with $\varepsilon = h(A, \bar{\Delta})$. In general $\bar{\Delta}$ is not obliged to coincide with Δ_0.*

All this follows from Construction 2.4 and the next well-known theorem about dynamical systems in S (see, for instance, [Ni]).

THEOREM 2.7. *Let T: $S \to S$ be an order preserving homeomorphism of S onto itself. Let there exist some $e_0 \in S$ for which the sequence $\{T^n e_0\}_{n \geqslant 0}$ is periodical with minimal period k. Then for every $e \in S$ the sequence $\{T^n e\}_{n \geqslant 0}$ is asymptotically k-periodical, i.e., $\{T^{nk} e\}_{n \geqslant 0}$ converges to some $e^* \in S$. In particular, $T^k e^* = e^*$ and the sequence $\{T^n e^*\}_{n \geqslant 0}$ is k-periodical.*

It should be noted also that for our approximation problem only those k-periodical sequences $\{T^n(e, \varepsilon)\}_{n \geqslant 0}$ are of interest for which just one turn around S is done for k iterations. If more turns around S are made for k iterations, then the corresponding k-gon is not convex.

We will need one more thing from the dynamical system theory. Denote by $\alpha(e)$ the length of the arc $[e, T(e, \varepsilon)]$ and put for $k = 1, 2, \ldots$

$$f^k(e, \varepsilon) := \alpha(e) + \alpha(T(e, \varepsilon)) + \cdots + \alpha(T^{k-1}(e, \varepsilon)).$$

It is known (see [Ni]) that the sequence $\{f^k(e, \varepsilon)/k\}_{k \geqslant 1}$ is convergent for every $e \in S$ and its limit does not depend on $e \in S$.

DEFINITION 2.8. The number

$$r(\varepsilon) = \frac{1}{2\pi} \lim_{k \to \infty} \frac{f^k(e, \varepsilon)}{k}$$

is called a rotation number of $T(\cdot, \varepsilon)$.

It is known (see [Ni]) that $r(\varepsilon) = p/q$, for some positive integers p and q without common divisors, if and only if for at least one $e \in S$ the sequence $\{T^n(e, \varepsilon)\}_{n \geqslant 0}$ is q-periodical and $f^q(e, \varepsilon) = 2p\pi$ (i.e., for q iterations of e by T the length of S is run over p times in the counterclockwise direction).

COROLLARY 2.9. *Let $\varepsilon = h(A, \Delta)$, where the convex polygon Δ is alternating for A. Then $r(\varepsilon) = 1/k$ if and only if Δ is a (nondegenerate) k-gon.*

Proof. Starting from a side direction of Δ one gets the periodical sequence $\{T^n(e, \varepsilon)\}_{n \geqslant 0}$ consisting of all side directions of Δ. Moreover, one turn around S is done for k iterations, where k is the number of different side directions of Δ. This is the case if and only if $r(\varepsilon) = 1/k$. ∎

PROPOSITION 2.10. *The function $r(\varepsilon)$ is nondecreasing and continuous.*

Proof. The continuity of $r(\varepsilon)$ follows from more general considerations but we give here a self-contained proof. The monotonicity of $f^k(e, \varepsilon)$ (as a function of $\varepsilon > 0$) follows from (ii) and (iii) of Proposition 2.5. This implies that $r(\varepsilon)$ is a nondecreasing function of ε. Such a function is discontinuous at some ε_0 if and only if

$$\lim_{\varepsilon \to \varepsilon_0,\, \varepsilon < \varepsilon_0} r(\varepsilon) < \lim_{\varepsilon \to \varepsilon_0,\, \varepsilon > \varepsilon_0} r(\varepsilon)$$

(i.e., $r(\varepsilon)$ has a jump at ε_0). Therefore, the rest of the proof is contained in the next assertion. ∎

LEMMA 2.11. *Let $r(0) := \lim_{\varepsilon \to 0} r(\varepsilon)$ and $r(\infty) := \lim_{\varepsilon \to \infty} r(\varepsilon)$. For every rational number p/q with $r(0) < p/q < r(\infty)$ there exists an $\varepsilon > 0$ such that $r(\varepsilon) = p/q$.*

Proof. There exist $\varepsilon_2 > \varepsilon_1 > 0$ such that $r(\varepsilon_1) < p/q < r(\varepsilon_2)$. If $s = nq$, then

$$\frac{1}{s} f^s(e, \varepsilon_1) = \frac{1}{nq} f^{nq}(e, \varepsilon_1) = \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{q} f^q(T^{iq}(e, \varepsilon_1), \varepsilon_1).$$

When $s = nq$ is large enough, $(1/2\pi)(1/nq) f^{nq}(e, \varepsilon_1) < (p/q)$. On the other hand, since $(1/2\pi)(1/nq) f^{nq}(e, \varepsilon_1)$ is a mean value of the numbers $(1/2\pi)(1/q) f^q(T^{iq}(e, \varepsilon_1), \varepsilon_1)$, $i = 0, 1, ..., n-1$, at least one of them is smaller than p/q. This means that $e_1 \in S$ exists for which $(1/2\pi)(1/q) f^q(e_1, \varepsilon_1) < (p/q)$. Reasoning in an analogous way we deduce that there exists some $e_2 \in S$ for which $(1/2\pi)(1/q) f^q(e_2, \varepsilon_2) > (p/q)$. Hence $f^q(e_1, \varepsilon_1) < 2p\pi < f^q(e_2, \varepsilon_2)$. Since $f^q(\cdot, \cdot)$ is a continuous function and its domain $S \times R^+$ is a connected set, there exist $\bar{e} \in S$ and $\bar{\varepsilon} > 0$ such that $f^q(\bar{e}, \bar{\varepsilon}) = 2p\pi$. Evidently $f^q(T^{iq}(\bar{e}, \bar{\varepsilon}), \bar{\varepsilon}) = 2p\pi$. Therefore

$$r(\bar{\varepsilon}) = \frac{1}{2\pi} \lim_{n \to \infty} \frac{1}{nq} f^{nq}(\bar{e}, \bar{\varepsilon}) = \frac{1}{2\pi} \lim_{n \to \infty} \frac{1}{nq} \sum_{i=0}^{n-1} f^q(T^{iq}(\bar{e}, \bar{\varepsilon}), \bar{\varepsilon}) = \frac{p}{q}. \quad \blacksquare$$

COROLLARY 2.12. *For every integer $k > 1$ the set $\{\varepsilon > 0 : r(\varepsilon) = 1/k\}$ is a (possibly degenerate) segment which is a closed subset of R^+.*

In Fig. 2.4 the graphs of $r(\varepsilon)$ are plotted for different sets A.

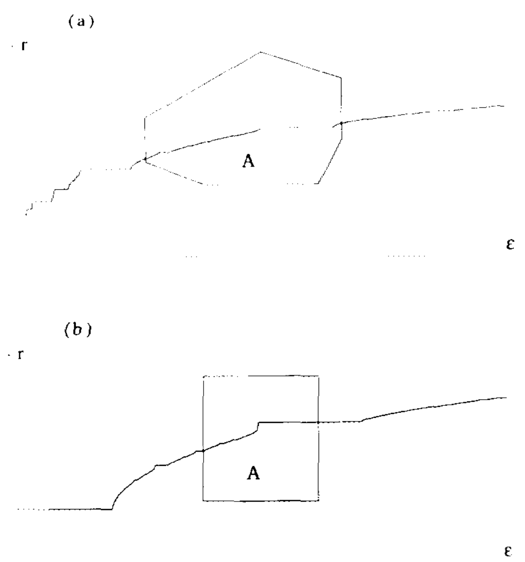

FIGURE 2.4

3. Algorithms and Numerical Results

Later in this section we will present an algorithm for the numerical computation of $T(e, \varepsilon)$. Now we will show how the possibility to calculate $T(e, \varepsilon)$ can be used in order to solve our problem of polygonal approximation of some $A \in \text{CONV}$. When $\varepsilon > 0$ and $e \in S$ are given, there is an attractive way to find a polygon Δ which is alternating for A and $h(A, \Delta) = \varepsilon$. One has to look at the behavior of the sequence $\{T^n(e, \varepsilon)\}_{n \geqslant 0}$. If it is recognized as "asymptotically k-periodical," then a k-gon Δ with the desired properties can be constructed because we will know its side directions. Unfortunately, Corollary 2.6 and Theorem 2.7 do not give any estimates for the rate of convergence of the sequence $\{T^{nk}(e, \varepsilon)\}_{n \geqslant 0}$ and it is not easy to forecast the number of iterations needed to reveal the periodicity of the sequence $\{T^n(e, \varepsilon)\}_{n \geqslant 0}$. Numerical experiments show, however, that 20–30 iterations of T suffice to establish the asymptotic k-periodicity of that sequence (at least for $k = 3$, $k = 4$). In particular, if the number $\varepsilon_k^* = \min\{h(A, \Delta): \Delta \in \text{POLY}_k\}$ is taken as ε, then after relatively few iterations the sequence $\{T^n(e, \varepsilon)\}_{n \geqslant 0}$ will provide the side direction of a k-gon Δ_0 which will be a best Hausdorff approximation for A in POLY_k. This remark reduces our approximation problem to the finding of ε_k^*. By Theorem 2.2 $\varepsilon_k^* = \min\{h(A, \Delta)$: $\Delta \in \text{POLY}_k$ and Δ is alternating for $A\}$. Having in mind Corollary 2.9 we see that $\varepsilon_k^* = \min\{\varepsilon > 0: r(\varepsilon) = 1/k\}$. The numerical procedure for the calculation of ε_k^* is now suggested by Proposition 2.10.

3.1. *An Algorithm for the Calculation of* ε_k^*.

If there were an efficient algorithm for the calculation of $r(\varepsilon)$, for every $\varepsilon \in R^+$, then solving an equation of the type $r(\varepsilon) = t$, where $t \in R$, would not be a difficult problem at all. For instance, one could use the bisection method in order to localize ε_k^*. Unfortunately the definition of $r(\varepsilon)$ as $\lim_{n \to \infty} (f^n(e, \varepsilon)/2\pi n)$ does not provide a good tool for the numerical calculation of $r(\varepsilon)$. Nevertheless we can use the specific features of our situation in order to find a good approximation for ε_k^*. In fact, it is enough to be able to estimate (for every $\varepsilon > 0$) the sign of the difference $r(\varepsilon) - 1/k$ in order to find a better approximation for ε_k^*. Starting from some $\varepsilon^0 > 0$ we can take as a next approximation some $\varepsilon^1 > \varepsilon^0$ (if $r(\varepsilon^0) - 1/k < 0$) or some $\varepsilon^1 < \varepsilon^0$ (if $r(\varepsilon^0) - 1/k > 0$). Technically this can be done in different ways. Here is one of the possibilities. Starting from some $e \in S$ produce a prescribed (but big enough) number s of members of the sequence

$$e,\ T(e, \varepsilon^0),\ T^2(e, \varepsilon^0),\ \ldots,\ T^s(e, \varepsilon^0),\ \ldots,\ T^{s+k}(e, \varepsilon^0),\ \ldots$$

and calculate the number $\beta = \sum_{i=1}^{k} \alpha(T^{s+i}(e, \varepsilon^0))$. If $\beta > 2\pi$, according to Proposition 2.5 there exists $\varepsilon < \varepsilon^0$ for which

$$\sum_{i=1}^{k} \alpha(T^i(e_1, \varepsilon)) = 2\pi,$$

where $e_1 = T^s(e, \varepsilon^0)$. This means that $r(\varepsilon) = 1/k$ and thus $\varepsilon_k^* \leqslant \varepsilon < \varepsilon^0$. Therefore some $\varepsilon^1 < \varepsilon^0$ can be taken as a next approximation of ε_k^*. If $\beta < 2\pi$, we can take some $\varepsilon^1 > \varepsilon^0$ and repeat the procedure with ε^1 at the place of ε^0. If $\beta = 2\pi$, we have $\varepsilon_k^* \leqslant \varepsilon^0$ and again $\varepsilon^1 < \varepsilon^0$ should be taken as next approximation of ε_k^*.

3.2. *An Algorithm for the Calculation of* $T(e, \varepsilon)$.

Now we describe an algorithm for the calculation of $T(e, \varepsilon)$. As mentioned in the introduction, the set A is completely determined by its support function $s_A : S \to R$. We will assume, however, that something more is known about the set A. Namely, we will consider that for every $e \in S$ at least one point $P(e) = (x(e), y(e))$ from the set $\{P \in A : \langle e, P \rangle = s_A(e)\}$ is known. $P(e)$ is a point where the function $\langle e, \cdot \rangle$ attains its maximum on A. To find $T(e, \varepsilon)$ one proceeds as follows:

(1) Find an $e^* \in [e, -e]$ such that $\langle e, P(\bar{e}) + \varepsilon\bar{e} \rangle < s_A(e) - \varepsilon$, for every $\bar{e} \in [e, e^*]$ and $\langle e, P(\bar{e}) + \varepsilon\bar{e} \rangle > s_A(e) - \varepsilon$, for every $\bar{e} \in (e^*, -e]$. In other words, e^* splits the arc $[e, -e]$ into two disjoint sets of points. The points $P(\bar{e}) + \varepsilon\bar{e}$ are on one side of the line $L = \{P \in R^2 : \langle e, P \rangle = s_A(e) - \varepsilon\}$ for $\bar{e} \in [e, e^*)$ and on the other side of this line for $\bar{e} \in (e^*, -e]$. The unit vector e^* can be found with satisfying accuracy by bisection method. There occur only two possibilities.

(a) There exists (see Fig. 3.1(a)) only one point $P \in A$ for which $\langle e^*, P \rangle = s_A(e^*)$. In this case $P = P(e^*)$ and we put $M := P(e^*) + \varepsilon e^*$ and proceed to step 2.

(b) There are (see Fig. 3.1(b)) at least two points P (and thus a line segment of points P on the boundary) of A for which $\langle e^*, P \rangle = s_A(e^*)$. $P(e^*)$ is among these points. In this case we take a unit vector e'' with $\langle e'', e^* \rangle = 0$ and find $t \in R$ in such a way that $\langle e, P(e^*) + \varepsilon e^* + te'' \rangle = s_A(e) - \varepsilon$. Then we put $M := P(e^*) + \varepsilon e^* + te''$ and proceed to step 2.

(2) Find $e' \in [e^*, -e^*]$ such that $\langle e', M \rangle = s_A(e') - \varepsilon$. This is possible because (according to Proposition 2.1 of [Ke]) the function $\langle \bar{e}, M \rangle - s_A(\bar{e})$ is strictly increasing from $-\varepsilon$ (for $\bar{e} = e^*$) to ε (for $\bar{e} = e'$) and takes values greater than ε in the open arc $(e', -e^*)$. Again a bisectional procedure could be applied.

Finally we put $T(e, \varepsilon) := e'$.

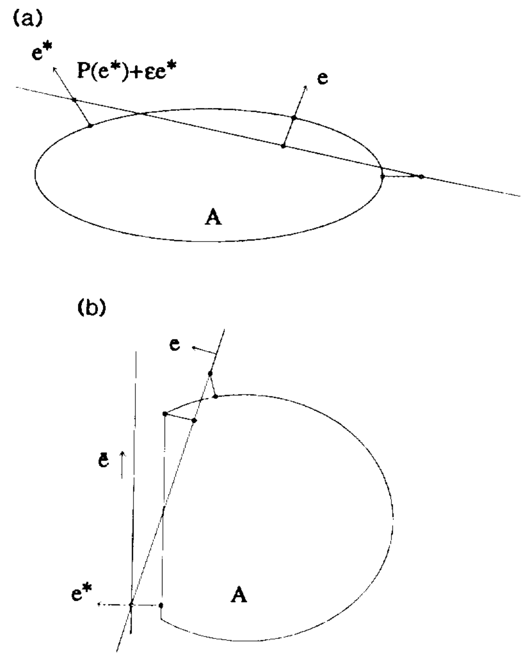

FIGURE 3.1

3.3. *An Algorithm for the Construction of $\Delta_n(e)$ (see Theorem* 2.3.)

According to Proposition 2.5 the function

$$f^n(e, \varepsilon) = \alpha(e) + \alpha(T(e, \varepsilon)) + \cdots + \alpha(T^{n-1}(e, \varepsilon))$$

is an increasing and continuous function of ε. For $\varepsilon_0 = 0.5\ W(e)$, where $W(e) = s_A(e) - s_A(-e) = \max\{\langle e, X\rangle : X \in A\} - \min\{\langle e, X\rangle : X \in A\}$ is the width of A in direction e, $T(e, \varepsilon_0) = -e$ and $T(-e, \varepsilon_0) = e$. Therefore $f^n(e, \varepsilon_0) = n\pi$. Consequently, if $f^n(e, 0) := \lim_{\varepsilon \to 0} f^n(e, \varepsilon) < 2\pi$ (this is the case when $A \notin \mathrm{POLY}_n$), then there is some ε for which $f^n(e, \varepsilon) = 2\pi$. This ε can be calculated by the bisection method. Evidently, e, $T(e, \varepsilon)$, ..., $T^{n-1}(e, \varepsilon)$ are the side directions of $\Delta_n(e)$ and the latter can be constructed effectively.

On the basis of these algorithms a computer program for the IBM PC/AT was developed which finds numerically the best approximation by n-gons of a given $A \in \mathrm{CONV}$.

In Fig. 3.2 some convex sets are shown together with their best approximating n-gons.

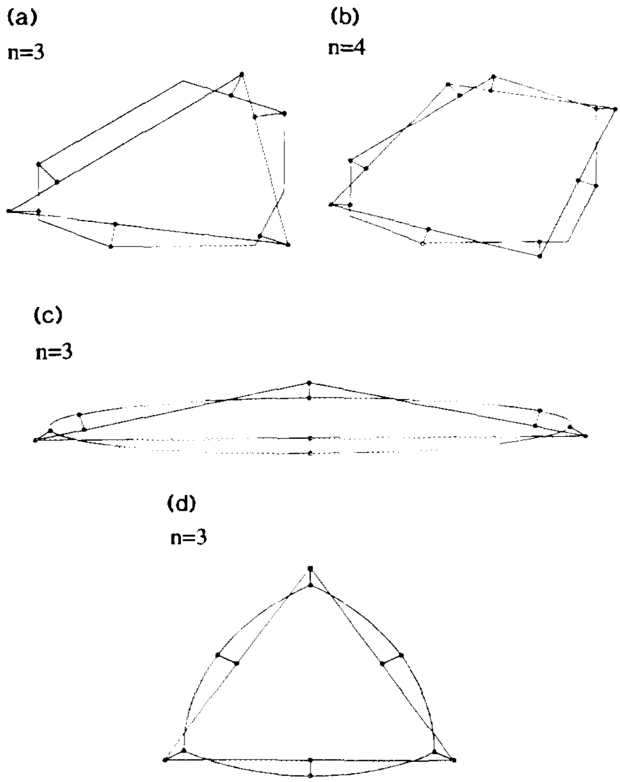

FIGURE 3.2

3.4 *Remarks*

Let $A \in \mathrm{CONV}$. If follows from Proposition 2.10 that there must exist real numbers $\varepsilon > 0$ for which the corresponding rotation number $r(\varepsilon)$ is an irrational number. For such an $\varepsilon > 0$ the sequence $\{T^i(e, \varepsilon)\}_{i \geqslant 0}$ does not "converge asymptotically" to a periodical sequence. Moreover (see [BK]), in this case, the sequence $\{T^i(e, \varepsilon)\}_{i \geqslant 0}$ is dense in S (for every $e \in S$). Further, if γ is an irrational number, then the set $\{\varepsilon > 0 : r(\varepsilon) = \gamma\}$ cannot have more than one point. This result from [BK] combined with Proposition 2.10 implies that the set $\{\varepsilon > 0 : r(\varepsilon)$ is rational$\}$ is dense in R^+. An expression of these facts for the case when A is a square can be seen on Fig. 3.3(a). It represents a copy of the computer monitor. The vertical coordinate line consists of the segment $[0, 2\pi]$ which is identified with S (in particular, $T^i(e, \varepsilon) \in [0, 2\pi]$ for $i = 1, 2, 3, \ldots$). The horizontal coordinate line consists of positive numbers. For fixed $\varepsilon > 0$ and $e \in S \equiv [0, 2\pi]$ the points $\{(\varepsilon, T^i(e, \varepsilon))\}_{i=100}^{1000}$ are displayed on the computer monitor as points on the vertical line passing trough ε. If for the first 100 "hidden" iterations the sequence $\{T^i(e, \varepsilon)\}_{i \geqslant 0}$ "stabilizes" and "becomes k-periodical," then the next 900 iterations will produce cyclically (on the computer monitor) only k points above the given $\varepsilon > 0$. For such ε the rotation number $r(\varepsilon)$ is rational. If $r(\varepsilon)$ is irrational for some $\varepsilon > 0$, then the points $\{(\varepsilon, T^i(e, \varepsilon))\}_{i=100}^{1000}$ appear above ε without any periodicity.

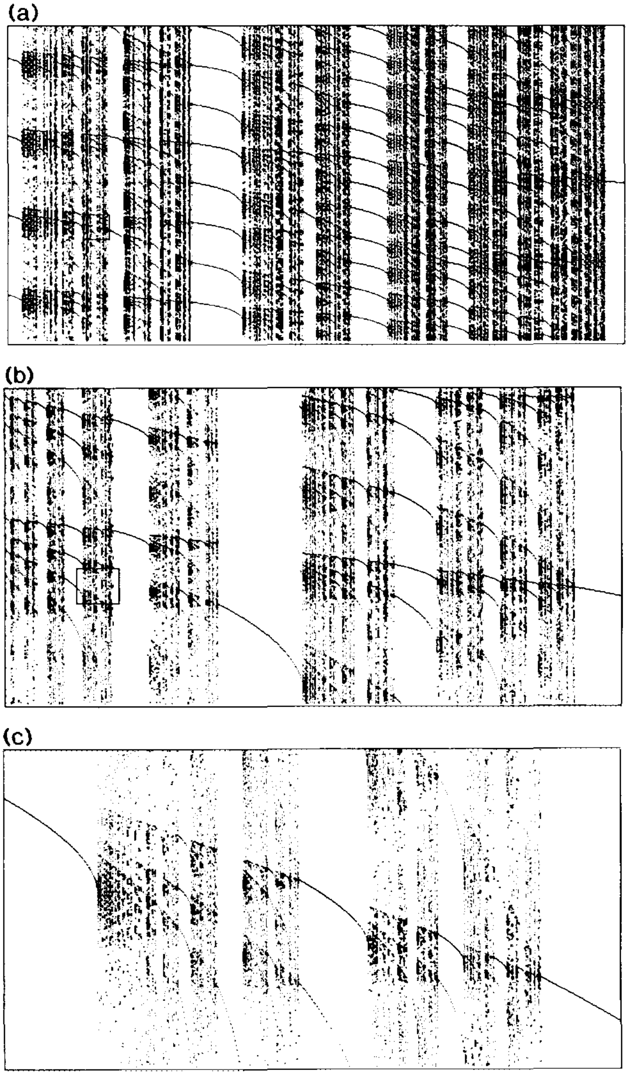

FIGURE 3.3

Figure 3.3(a) has a certain degree of self-similarity. When enlarged to the full size of the monitor the small rectangle depicted on 3.3(a) transforms to what is shown on 3.3(b). In turn, 3.3(c) is an enlargement of the small rectangle from 3.3(b).

Another result related to our work is given in [Ne]. If two convex sets A_1 and A_2 give rise (via Construction 2.4) to one and the same mapping $T(\cdot, \cdot): S \times R^+ \to S$, then A_1 and A_2 coincide up to a translation. This is proved in [Ne] for the cases when A_1 and A_2 are either convex polygons or smooth convex sets. One could conjecture that the result is valid for arbitrary convex sets.

Also, it is not known to what extent the two sets A_1 and A_2 coincide if $r_1(\varepsilon) = r_2(\varepsilon)$ for every $\varepsilon > 0$, where $r_i(\varepsilon)$ is the rotation number function of A_i, $i = 1, 2$.

Acknowledgment

The authors are indebted to L. Stoyanov for useful discussions.

References

[BK] E. Belogai and P. S. Kenderov, Dynamical systems associated with a convex set in the plane, preprint.

[Ge] P. G. Georgiev, Approximation of convex n-gons by $(n-1)$-gons, *in* "Proceeding of the Thirteenth Spring Conference of the Union of Bulgarian Mathematicians (1984)," pp. 289–303. [Bulgarian]

[GK] P. Gruber and P. Kenderov, Approximation of convex bodies by polytops, *Rend. Circ. Mat. Palermo (2)* **31** (1982), 195–225.

[Ke] P. S. Kenderov, Polygonal approximation of plane convex compacts, *J. Approx. Theory* **38** (1983), 221–239.

[McV] E. D. McClure and R. A. Vitale, Polygonal approximation of plane convex bodies, *J. Math. Anal. Appl.* **51** (1975), 326–358.

[Ne] M. Nedelcheva, Characterization of convex subsets of the plane through their local approximation properties, *Serdica* **11** (1985), 165–170.

[Ni] Z. Nitecki, "Differentiable Dynamics," MIT Press, Cambridge, MA, 1971.

[P] V. Popov, Approximation of convex sets, *Bull. Inst. Math. Bulgar. Acad. Sci.* **11** (1970), 67–79. [Bulgarian]

[T] F. Toth, Approximation by polygons and polyhedra, *Bull. Amer. Math. Soc.* **54** (1948), 431–438.

[YC] Y. Yotov and K. M. Christov, System for the construction of a m-polygon for the best approximation of a convex n-polygon, *in* "Proceeding of the Fourteenth Spring Conference of the Union of Bulgarian Mathematicians (1985)," pp. 412–421. [Bulgarian]